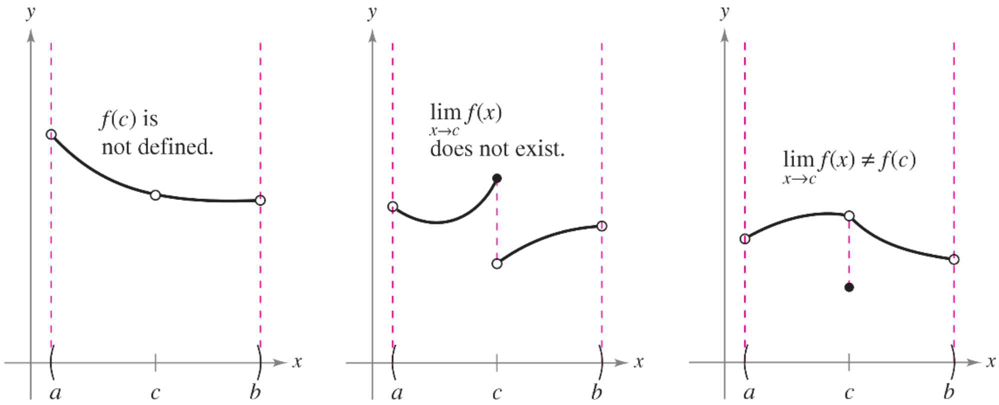
Continuty 连续 (P70)

Three conditions of not continuous at $x = c$

1. The function is not defined at $x = c$
2. The limit of $f(x)$ does not exist at $x = c$
3. The limit of $f(x)$ exist at $x = c$, but it is not equal to $f(c)$



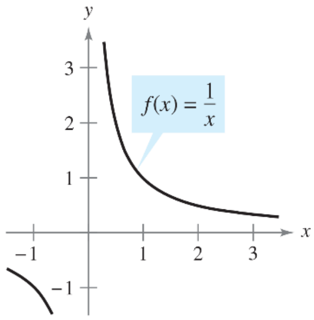
Continuity at a Point: (函数在一点连续)

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

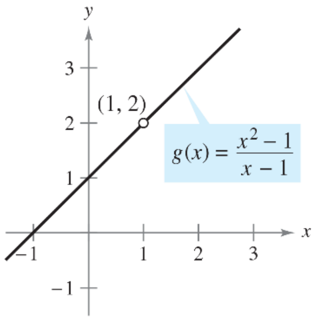
(开区间$(\boldsymbol{a}, \boldsymbol{b})$上连续)A function is **continuous on an open interval** (a, b) if it is continuous at each point in the interval.

(实数域上处处连续)A function that is continuous on the entire real line $(-\infty, \infty)$ is everywhere continuous.

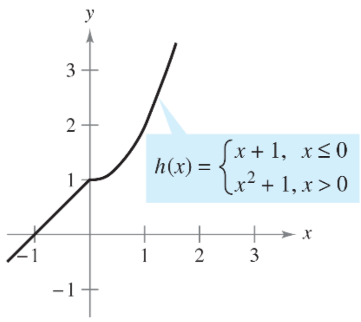
Example 1: Discuss the continuity of each function (p71)



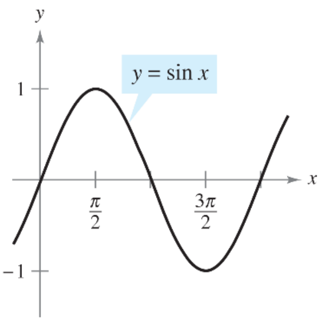
(a) Nonremovable discontinuity at $x = 0$



(b) Removable discontinuity at $x = 1$



(c) Continuous on entire real line



(d) Continuous on entire real line

a. $f(x) = \frac{1}{x}$

b. $g(x) = \frac{x^2-1}{x-1}$

c. $h(x) = \begin{cases} x+1, & (x \le 0) \\ x^2+1, & (x > 0) \end{cases}$

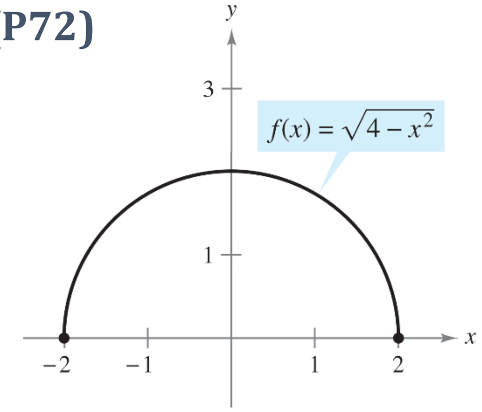
d. $y = \sin x$

Properties of continuity 连续的性质

b is a real number, $f(x)$ and $g(x)$ are continuous at $x = c$:

1. Scalar multiple: $bf(x)$
2. Sum or difference: $f(x) \pm g(x)$
3. Product: $f(x)g(x)$
4. Quotient: $f(x)/g(x)$, if $g(x) \neq 0$
5. Composite: $f(g(x))$

One-side limit 单边极限 (P72)



$f(x)=\sqrt{4-x^2}$

$\lim_{x\to 2^-}\sqrt{4-x^2}=0$

is a one-side limit

The existance of a limit 极限存在定理 (P73)

The limit of $f(x)$ as x approaches c is L $\Longleftrightarrow \begin{cases}\lim_{x\to c^-} f(x)=L\\ \lim_{x\to c^+} f(x)=L\end{cases}$

极限存在⇔左极限=右极限

(闭区间[a,b]上连续)A function is **continuous on the closed interval**

$[a,b]$ if it continuous on the open interval (a,b) and $\lim_{x\to a^+} f(x)=f(a)$

and $\lim_{x\to B^-} f(x)=f(b)$.

The intermediate value theorem 中值存在定理

(P77) If $f(x)$ is continuous on the closed interval $[a,b]$, $f(x)\neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c)=k$.

Application: the bisection method (二分法)

Exercise1: Find the following limit (if it exists). Explain your reasoning.

$\lim_{x\to 3} f(x), where\ f(x) = \begin{cases} (x-3)^2, & x \le 3 \\ 3-x, & x > 3 \end{cases}$

方法：在 3 这点，左极限=右极限=0，所以连续

Exercise2: Find all values of c such that $f(x)$ is continuous on $(-\infty, \infty)$.

$f(x) = \begin{cases} 1-x^2, & x \le c \\ x, & x > c \end{cases}$

方法：$f1(x) = 1 - x^2$，$f2(x) = x$. 若要求$f(x)$连续，必须$f1(x) = f2(x)$，即 $1 - x^2 = x$，解得$c = x = (-1 \pm \sqrt{5})/2$

Exercise3: Find the constant a such that the function is continuous on the entire real line. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

方法：当 $x = 1$时 $3x^2 = 3(1)^2 = 3$，所以$ax - 4 = a(1) - 4 = 3$，所以$a = 7$.

Exercise4: Explain why $f(x) = 3x^2 - 4$ is guaranteed to have a 0 in the given interval $[1,2]$

方法：$\left.\begin{matrix} \lim_{x\to c} f(x) = f(c), 所以f(x)处处连续 \\ f(1) = -1 < 0, f(2) = 8 > 0 \end{matrix}\right\} \xRightarrow{极限存在定理} f(x)$在$[1,2]$上连续

Exercise1: Find the following limit (if it exists). Explain your reasoning.

$\lim_{x \to 3} f(x), where\ f(x) = \begin{cases} (x-3)^2, & x \le 3 \\ 3-x, & x > 3 \end{cases}$

Exercise2: Find all values of c such that $f(x)$ is continuous on $(-\infty, \infty)$.

$f(x) = \begin{cases} 1-x^2, & x \le c \\ x, & x > c \end{cases}$

Exercise3: Find the constant a such that the function is continuous on the entire real line. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax-4, & x < 1 \end{cases}$

Exercise4: Explain why $f(x) = 3x^2 - 4$ is guaranteed to have a 0 in the given interval [1,2]

Page 86, Peterson's Master AP Calculus AB&BC, 2nd Edition, W. Michael Kelley.

EXERCISE 3

Directions: Solve each of the following problems. Decide which is the best of the choices given and indicate your responses in the book.

DO *NOT* USE A GRAPHING CALCULATOR FOR ANY OF THESE PROBLEMS.

For problems 1 through 3, explain why each function is discontinuous and determine if the discontinuity is removable or nonremovable.

1. $g(x) = \begin{cases} 2x-3, & x<3 \\ -x+5, & x \geq 3 \end{cases}$

2. $b(x) = \dfrac{x(3x+1)}{3x^2-5x-2}$

3. $h(x) = \dfrac{\sqrt{x^2-10x+25}}{x-5}$

4. Describe the continuity of the 15 functions you were to memorize in Chapter 2 without consulting any notes.

5. Draw the graph of a function, $f(x)$, that satisfies each of the following conditions. Then, describe the continuity of the function:

- $\lim\limits_{x \to 2} f(x) = -1$
- $\lim\limits_{x \to 0^+} f(x) = -\infty$
- $\lim\limits_{x \to 0^-} f(x) = \infty$
- $f(2) = 4$
- $f(-1) = f(3) = 0$
- f increases on its entire domain

6. Find the value of k that makes p continuous if $p(x) = \begin{cases} -|x-2|+k, & x \leq 4 \\ -x^2+11x-23, & x>4 \end{cases}$.

ANSWERS AND EXPLANATIONS

1. g is made up of two polynomial (linear) segments, both of which will be continuous everywhere. However, the graph has a jump discontinuity at $x = 3$. Notice that $\lim\limits_{x \to 3^-} g(x) = 3$ (you get this by plugging 3 into the $x < 3$ rule). The right-hand limit of $g(x)$ at $x = 3$ is 2. Because the left- and right-hand limits are unequal, no general limit exists at $x = 3$, breaking the first condition of continuity. Furthermore, because no limit exists, the discontinuity is nonremovable.

2. Because b is rational, b will be continuous for all x in the domain. However, $x = -\dfrac{1}{3}$ and 2 are not in the domain. Using the factoring method of evaluating limits, you get that $\lim\limits_{x \to -\frac{1}{3}} \dfrac{x}{x-2} = \dfrac{1}{7}$, so $x = -\dfrac{1}{3}$ is a removable discontinuity. No limit exists at $x = 2$, an essential discontinuity.

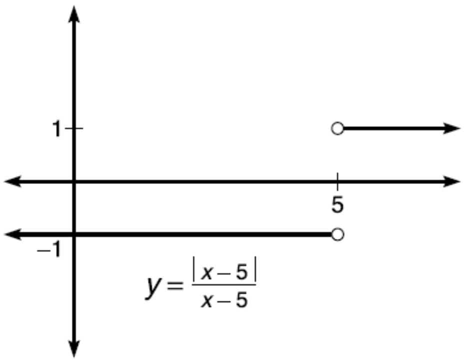
3. The numerator of the fraction is a perfect square, so simplify to get

$$h(x) = \frac{\sqrt{(x-5)^2}}{x-5}$$

Remember that the square root function has a positive range, so the numerator must be positive:

$$h(x) = \frac{|x-5|}{x-5}$$

It helps to think about this graphically. After substituting some values of x into h, you get the following graph:



Thus, h has a jump discontinuity at $x = 5$.

4. $y = x$: continuous on $(-\infty,\infty)$

$y = x^2$: continuous on $(-\infty,\infty)$

$y = x^3$: continuous on $(-\infty,\infty)$

$y = \sqrt{x}$: continuous on $[0, \infty)$

$y = |x|$: continuous on $(-\infty,\infty)$

$y = \frac{1}{x}$: continuous on $(-\infty,0) \cup (0,\infty)$

$y = [[x]]$: continuous for all real numbers x, if x is not an integer

$y = e^x$: continuous on $(-\infty,\infty)$

$y = \ln x$: continuous on $(0,\infty)$

$y = \sin x$: continuous on $(-\infty,\infty)$

$y = \cos x$: continuous on $(-\infty,\infty)$

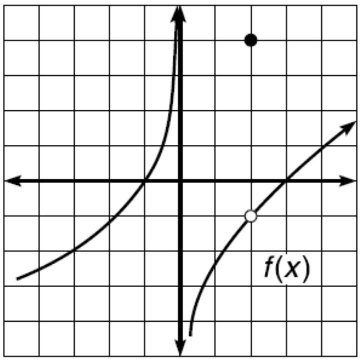
$y = \tan x$: continuous for all real numbers x, if $x \neq \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ (which can also be written $x \neq 2(n+1)\cdot\frac{\pi}{2}$, when n is an integer)

$y = \cot x$: continuous for all real numbers x if $x \neq \ldots, -\pi, 0, \pi, \ldots$ (which can also be written $x \neq n\pi$, when n is an integer)

$y = \sec x$: continuous for all real numbers x, $x \neq 2(n+1)\cdot\frac{\pi}{2}$, when n is an integer

$y = \csc x$: continuous for all real numbers x, $x \neq n\pi$, when n is an integer

5. There is some variation in the possible answer graphs, but your graph should match relatively closely.



6. Just like Number 1 in this problem set, the $x \le 4$ rule evaluated at 4 represents $\lim_{x \to 4^-}$, and the $x > 4$ rule evaluated at 4 represents $\lim_{x \to 4^+}$. In order for p to be continuous, these limits must be equal.

$$\lim_{x \to 4^+} p(x) = -(4)^2 + 11(4) - 23 = 5$$

$$\text{Therefore, } \lim_{x \to 4^-} p(x) = 5$$

$$\lim_{x \to 4^-} p(x) = -|4 - 2| + k = 5$$

$$-2 + k = 5$$

$$k = 7$$

HANDS-ON ACTIVITY 3.2: THE EXTREME VALUE THEOREM

This and the next activity introduce you to two basic but important continuity theorems. Note that both of these are called *existence theorems*. They guarantee the existence of certain values but do not tell you where these values are—it's up to you to find them, and they're always in the last place you look (with your car keys, your wallet from two years ago, and the words to that Bon Jovi song you used to know by heart).

1. Given $f(x) = x^4 - 3x - 4$, justify that $f(x)$ is continuous on the x interval $[-1,2]$.